Q1a

a) $z_2 = z_1^* = (5+i)^* = 5-i$

$$\boxed{z_2 = 5-i}$$

Therefore

$$(z-z_1)(z-z_2) = (z-(5+i))(z-(5-i))$$
$$= z^2 - 10z + 26$$

is a factor of $z^3 - 11z^2 + 36z - 26$

So

$$z^3 - 11z^2 + 36z - 26 = (z^2 - 10z + 26)(z-1)$$

That means

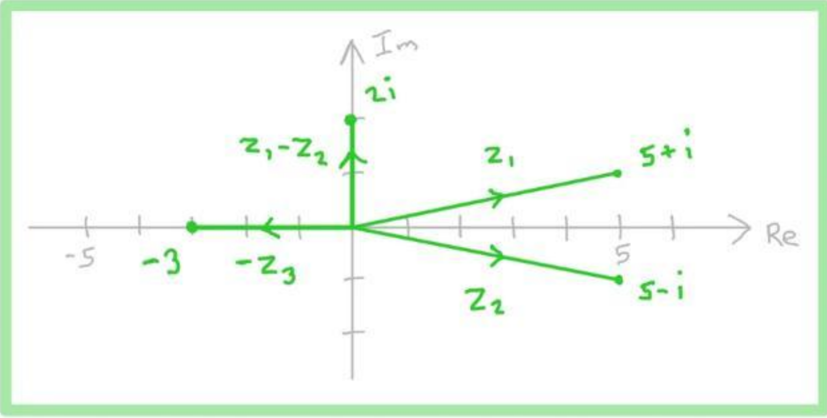
$$\boxed{z_3 = 1}$$

is the third solution.

Q1b

b) (i) $-3z_3 = -3(1) = -3$

$z_1 - z_2 = (5+i) - (5-i) = 2i$




(ii) Reflection in the real axis.

Q2

(i) $z^2 = \left(-2 + \frac{15}{4}i\right)^2 = 4 - \frac{15}{2}i - \frac{15}{2}i + \frac{225}{16}i^2$

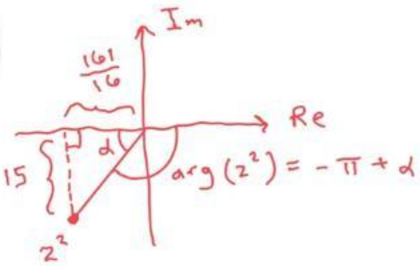
$i^2 = -1$

$$\boxed{z^2 = -\frac{161}{16} - 15i}$$

(ii) $|z^2| = \sqrt{\left(-\frac{161}{16}\right)^2 + (-15)^2}$

$$\boxed{|z^2| = \frac{289}{16}}$$

(iii)




$$\arg(z^2) = -\pi + \alpha$$
$$= -\pi + \tan^{-1}\left(\frac{15}{161/16}\right)$$
$$= -2.161678\ldots$$

$$\boxed{\arg(z^2) = -2.16 \ (2 \text{ d.p.})}$$

Q3a

a) $\frac{z_1}{z_2} = -1 + i$

$$z_2 = \frac{z_1}{-1+i} = \frac{5+pi}{-1+i}$$

$$z_2 = \frac{5+pi}{-1+i} \times \frac{-1-i}{-1-i}$$

$$= \frac{-5-5i-pi-pi^2}{1+i-i-i^2} \qquad i^2 = -1$$

$$= \frac{(p-5)+(-p-5)i}{2}$$

$$\boxed{z_2 = \frac{p-5}{2} - \frac{p+5}{2} i}$$

Q3b

From part (a),

$$z_2 = \frac{p-5}{2} - \frac{p+5}{2} i$$

$$|z| = |a+bi| = \sqrt{a^2+b^2}$$

b) $$|z_2| = \sqrt{\left(\frac{p-5}{2}\right)^2 + \left(-\frac{p+5}{2}\right)^2} = \sqrt{73}$$

So $$\left(\frac{p-5}{2}\right)^2 + \left(-\frac{p+5}{2}\right)^2 = 73$$

$$\frac{p^2-10p+25}{4} + \frac{p^2+10p+25}{4} = 73$$

$$p^2 - \cancel{10p} + 25 + p^2 + \cancel{10p} + 25 = 292$$

$$2p^2 + 50 = 292$$

$$2p^2 = 242$$

$$p^2 = 121$$

$$\boxed{p = 11 \text{ or } -11}$$

Q3C

From part (a),

$$z_2 = \frac{p-5}{2} - \frac{p+5}{2}i$$

From part (b),

$$p = 11 \text{ or } -11$$

c) <u>If $p = 11$</u>: $z_2 = \frac{11-5}{2} - \frac{11+5}{2}i = 3 - 8i$

$\arg(3-8i) = -\tan^{-1}\left(\frac{8}{3}\right)$ ⟵ in $4^{th}$ quadrant of Argand diagram

$= -1.212025\ldots$

<u>If $p = -11$</u>: $z_2 = \frac{-11-5}{2} - \frac{-11+5}{2}i = -8 + 3i$

$\arg(-8+3i) = \pi - \tan^{-1}\left(\frac{3}{8}\right)$ ⟵ in $2^{nd}$ quadrant of Argand diagram

$= 2.782821\ldots$

$= 2.78$ (2 d.p.)

So $p = -11$ and $z_2 = -8 + 3i$

$$\boxed{\operatorname{Im}(z_2) = 3}$$

<u>Alternative method without calculating arguments</u>

Note that $\frac{\pi}{2} < 2.78 < \pi$. So $\arg(z_2) = 2.78$ means that $z_2$ is in the $2^{nd}$ quadrant on an Argand diagram. Of the two possibilities, only $-8+3i$ is in the $2^{nd}$ quadrant.

So $z_2 = -8+3i$ and $\operatorname{Im}(z_2) = 3$.

Q4A

$r = |z|$

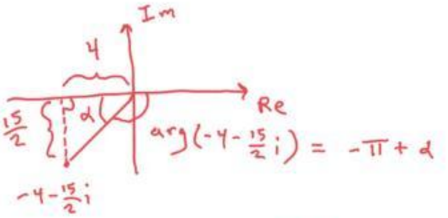
$z = r(\cos\theta + i\sin\theta)$

$\theta = \arg(z)$

$$|z| = |a+bi| = \sqrt{a^2+b^2}$$

a) $r = \left|-4-\frac{15}{2}i\right| = \sqrt{(-4)^2+\left(-\frac{15}{2}\right)^2} = \frac{17}{2}$

$\theta = \arg\left(-4-\frac{15}{2}i\right)$




$$\begin{aligned}\theta &= -\pi + \tan^{-1}\left(\frac{15/2}{4}\right)\\ &= -\pi + \tan^{-1}\left(\frac{15}{8}\right)\\ &= -2.060753\ldots\\ &= -2.06 \quad (2\text{ d.p.})\end{aligned}$$

$$\boxed{\frac{17}{2}\left(\cos(-2.06) + i\sin(-2.06)\right)}$$

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Q4B

b) $\sqrt{12}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$

$$\begin{aligned}&= \sqrt{12}\left(-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)\\ &= -\frac{\sqrt{12}\sqrt{3}}{2} - \frac{\sqrt{12}}{2}i\end{aligned}$$

$$\boxed{= -3-\sqrt{3}\,i}$$

Q5

For the general complex numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

given in modulus-argument form, use algebra and the appropriate trigonometric compound angle formulae to prove the results

$$|z_1z_2| = |z_1||z_2| \quad \text{and} \quad \arg(z_1z_2) = \arg(z_1) + \arg(z_2).$$

[4]

$r = |z|$

$z = r(\cos\theta + i\sin\theta)$

$\theta = \arg(z)$

$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$ [Compound angle formula]

$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$ [Compound angle formula]

$|z_1| = r_1 \quad \arg(z_1) = \theta_1 \quad |z_2| = r_2 \quad \arg(z_2) = \theta_2$

$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\,r_2(\cos\theta_2 + i\sin\theta_2)$ $\quad i^2 = -1$

$= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$

$= r_1r_2\big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big)$

$= r_1r_2\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big)$

Therefore

$|z_1z_2| = r_1r_2$ and $\arg(z_1z_2) = \theta_1 + \theta_2$

So

$$\boxed{|z_1z_2| = |z_1||z_2| \qquad \arg(z_1z_2) = \arg(z_1) + \arg(z_2)}$$

Q6

$z_1 = 9\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$z_2 = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

Work out

(i) $z_1z_2$

(ii) $\frac{z_1}{z_2}$

giving your answers in modulus-argument form with $\theta$ in the interval $-\pi < \theta \leq \pi$.

[4]

$r = |z|$

$z = r(\cos\theta + i\sin\theta)$

$\theta = \arg(z)$

$|z_1z_2| = |z_1||z_2| \quad \arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

(i) $|z_1z_2| = (9)(4) = 36$

$\arg(z_1z_2) = \frac{\pi}{6} + \frac{4\pi}{3} = \frac{3\pi}{2}$ outside range

$\frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$ in range

$$\boxed{z_1z_2 = 36\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)}$$

Note: $z_1z_2 = -36i$

(ii) $\left|\frac{z_1}{z_2}\right| = \frac{9}{4}$

$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{6} - \frac{4\pi}{3} = -\frac{7\pi}{6}$ outside range

$-\frac{7\pi}{6} + 2\pi = \frac{5\pi}{6}$ in range

$$\boxed{\frac{z_1}{z_2} = \frac{9}{4}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$$

Q7

Given that $|z + 1 - 2i| = |z - 7 + 4i|$ :

(i) On an Argand diagram, sketch the locus (i.e., set of points) for which the equation is true.

(ii) Shade the region of your diagram that satisfies the inequality $|z + 1 - 2i| > |z - 7 + 4i|$.

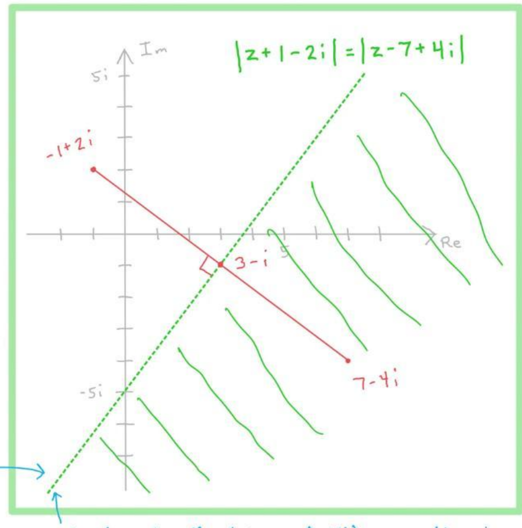
[4]

$|z-(-1+2i)| = |z-(7-4i)|$

$|z+1-2i|$ = distance from z to $-1+2i$ in Argand diagram

$|z-7+4i|$ = distance from z to $7-4i$ in Argand diagram

$|z+1-2i| = |z-7+4i|$ is the perpendicular bisector of the line segment connecting $-1+2i$ and $7-4i$




__Note__: This line has equation

$$y = \frac{4}{3}x - 5$$

but you don't need to give that to get the marks!

__Note__: 'greater than' in part (ii) means line is not included in the shaded region.

Q8a

(a) On an Argand diagram, sketch the loci (i.e., sets of points) for which each of the following equations is true:

(i) $\arg(z + 2 - 2i) = \frac{\pi}{4}$

(ii) $|z - 3 - 2i| = 5$

[4]

(b) Shade the region of your diagram that satisfies both of the following inequalities:

$$0 \leq \arg(z + 2 - 2i) \leq \frac{\pi}{4} \quad \text{and} \quad |z - 3 - 2i| \leq 5$$

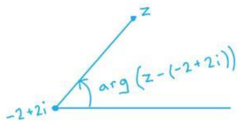
[2]

a) $\arg(z-(-2+2i)) = \frac{\pi}{4}$

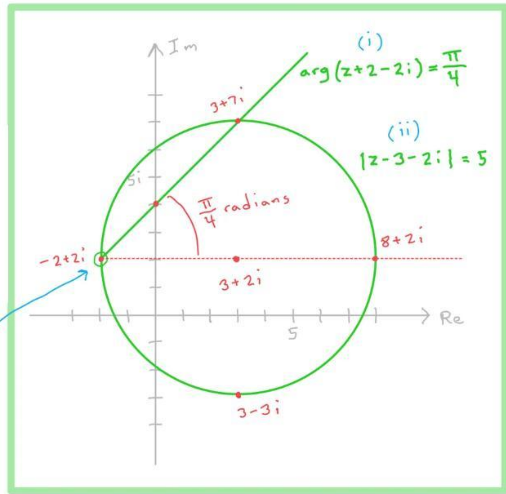
$|z-(3+2i)| = 5$

All the points at a distance of 5 from $3+2i$. I.e., circle radius 5, centre at $3+2i$.

$\arg(z-(-2+2i))$ is the angle that the line from $-2+2i$ to z makes with the line that starts at $-2+2i$ and goes off to the right parallel with the real axis. It is measured anticlockwise from the line parallel to the real axis.

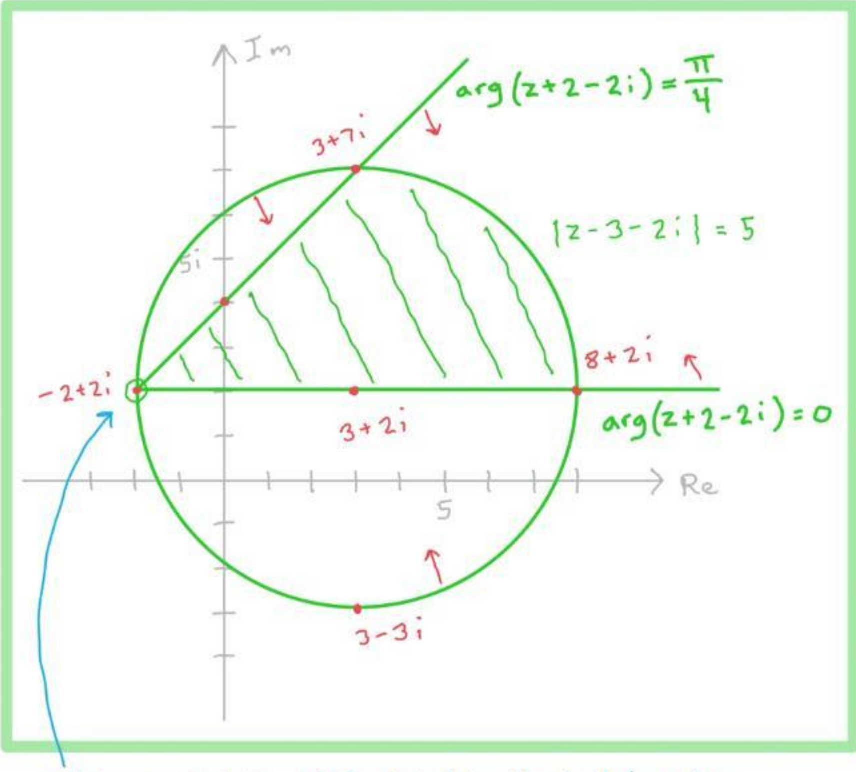



__Note__: $-2+2i$ is not included in the locus for $\arg(z+2-2i) = \frac{\pi}{4}$




Q8b

b)




Note: $-2+2i$ is not included in the shaded region, because $\arg((-2+2i)+2-2i)=\arg(0)$ is undefined.

Q9A

a) If $z=-7$ is a solution, then $(z+7)$ is a factor of the cubic:

$$z^3+9z^2+27z+91=(z+7)(z^2+bz+13)$$
$$=z^3+(b+7)z^2+(7b+13)z+91$$

Set $x^2$ coefficients equal:

$b+7=9 \Rightarrow b=2$

Check that $b=2$ works here too

$$z^3+9z^2+27z+91=(z+7)(z^2+2z+13)$$

The other two roots are solutions to

$$z^2+2z+13=0$$
$$(z+1)^2-1+13=0$$
$$(z+1)^2=-12$$
$$z+1=\pm\sqrt{-12}=\pm\sqrt{12}\,i=\pm 2\sqrt{3}\,i$$

Could also use quadratic formula here

$$\boxed{z=-1+2\sqrt{3}\,i \quad \text{or} \quad z=-1-2\sqrt{3}\,i}$$

Q9b

b)

$|(-7)+3| = |-4| = 4$

$|z| = |a+bi| = \sqrt{a^2+b^2}$

$|(-1+2\sqrt{3}i)+3| = |2+2\sqrt{3}i|$

$= \sqrt{2^2+(2\sqrt{3})^2}$

$= \sqrt{16} = 4$

$|(-1-2\sqrt{3}i)+3| = |2-2\sqrt{3}i|$

$= \sqrt{2^2+(-2\sqrt{3})^2}$

$= \sqrt{16} = 4$

$c = 4$

Q9C

c) $\underbrace{|z-(-3)| = 4}$

circle radius 4,
centre at -3